



MAS231 GEOMETRY II (B.Sc. examination by course units)

Duration: 2 hours

Date and time: To Be Determined

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions will be counted.

Question 1. (25 marks)

(a) (8 marks) Determine whether or not the knot diagram

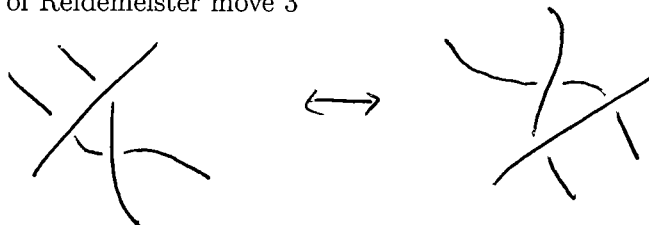


is tricolourable.

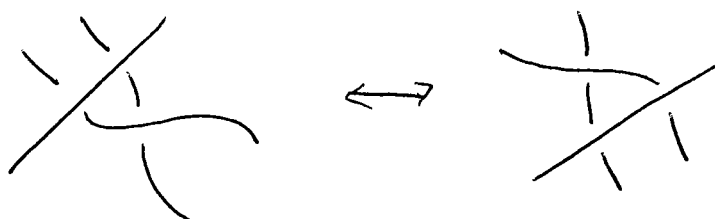
(b) (4 marks) Find the writhe of the knot diagram in (a).

(c) (5 marks) Prove that Reidemeister move 2 is invariant for writhe.

(d) (8 marks) Show that from Reidemeister move 1, Reidemeister move 2 and the following version of Reidemeister move 3

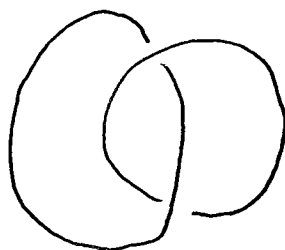


that this version of Reidemeister move 3 can be derived.

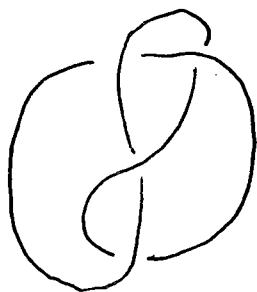


Question 2. (25 marks)

- (a) (3 marks) Define the Jones polynomial $J(t)$ of a knot diagram in terms of its writhe W and Kauffman Bracket. State what is the Jones polynomial $J_R(t)$ of the mirror image of a knot in terms of the Jones polynomial $J(t)$ of the knot itself.
- (b) (6 marks) Find the Kauffman Bracket of the Hopf link:



- (c) (7 marks) Determine the Kauffman Bracket of the figure-8 knot



You may use your answer to (b) and the fact that

$$B \left(\text{figure-8 knot} \right) = -X^5 - X^{-3} + X^{-7}$$

- (d) (6 marks) Determine the Jones polynomial $J(t)$ of the figure-8 knot.
- (e) (3 marks) Show that for the figure-8 knot that $J(t) = J_R(t)$.

Question 3. (25 marks)

- (a) (5 marks) Define what it means for a curve $\gamma(t)$ to be regular. Determine whether the curve $\gamma(t) = (\cos(t^2), \sin(t^2), t^2)$, $t \in (-\infty, \infty)$ is regular.
- (b) (9 marks) Define what it means for a curve $\tilde{\gamma}(\tilde{t})$, $\tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$ to be a reparametrization of a curve $\gamma(t)$, $t \in (\alpha, \beta)$ and prove that $\tilde{\gamma}(\tilde{t})$ is regular if and only if $\gamma(t)$ is regular.
- (c) (5 marks) State the definitions of the orthonormal triple τ , ν and β and state formulae for the derivatives $\dot{\tau}$, $\dot{\nu}$ and $\dot{\beta}$ involving the curvature K and the torsion T .
- (d) (6 marks) Consider the curve

$$\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}} \right), \quad 0 < t < 1/2.$$

Show that $\gamma(t)$ is unit-speed and compute the curvature K and the orthonormal triple τ , ν , β , for this curve.

Question 4. (25 marks)

- (a) (4 marks) Define signed curvature K_S and prove that $|K_S| = K$ where K is the usual, unsigned curvature.
- (b) Consider the curve

$$\gamma(t) = (t, \sqrt{t}, 0), \quad t \in (0, \infty).$$

- (i) (3 marks) Sketch $\gamma(t)$.
- (ii) (5 marks) Find $\tau(t)$, $\tau(1/4)$ and $\lim_{t \rightarrow \infty} \tau(t)$.
- (iii) (7 marks) Compute the signed curvature K_S for $\gamma(t)$ and use it to evaluate the integral

$$\int_{1/4}^{\infty} \frac{dt}{\sqrt{t}(4t+1)} dt.$$

- (c) (6 marks) Prove that if $\gamma(t)$ is a planar curve with torsion T , then $T = 0$.

Question 5. (25 marks)

Let S be the surface defined by

$$S = \{(x, y, z) : z = x^2 + y^2\}.$$

and consider the surface patches for S defined by

$$\sigma(u, v) = (u, v, u^2 + v^2), \quad u \in \mathbb{R}, v \in \mathbb{R}$$

and

$$\tilde{\sigma}(r, \theta) = (r \cos(\theta), r \sin(\theta), r^2), \quad r > 0, 0 < \theta < 2\pi$$

- (a) (5 marks) Find the unit normal N_σ for the surface patch σ and sketch the surface determined by σ indicating the direction of N_σ at a couple of typical points.
- (b) (6 marks) Find the unit normal $N_{\tilde{\sigma}}$ for the surface patch $\tilde{\sigma}$ and show that $N_{\tilde{\sigma}} = N_\sigma$ for values of u and v corresponding to r and θ .
- (c) (6 marks) Find the first and second fundamental forms for the surface patch $\tilde{\sigma}$.
- (d) (8 marks) Use your answer to (c) to determine the surface area of the part of S with $z \leq 2$.

Question 6. (25 marks) Let $\gamma(t) = \sigma(u(t), v(t))$ be a unit-speed curve on the surface patch

$$\sigma(u, v) = (u \cos(v), u \sin(v), v).$$

- (a) (4 marks) Compute the first fundamental form $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ for σ .
- (b) (3 marks) Show that $\dot{u}^2 + (1 + u^2)\dot{v}^2 = 1$, where $\dot{u} = \frac{d}{dt}u(t)$ and $\dot{v} = \frac{d}{dt}v(t)$.
- (c) (5 marks) Show that if γ is a geodesic on σ . Use the second geodesic equation

$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

to show that $\dot{v} = \frac{a}{1+u^2}$, where a is a constant.

- (d) (7 marks) Using the fact that $\frac{dv}{du} = \frac{\dot{v}}{\dot{u}}$, now show that

$$\left(\frac{dv}{du}\right)^2 = \frac{a^2}{(1 - a^2 + u^2)(1 + u^2)}.$$

- (e) (6 marks) Find the geodesics for $a = 0$ by solving the equation in (d) and show that if $a = 1$ then the functions $v = v_0 + \sinh^{-1}(1/u)$, v_0 a constant, solve the equation in (d).